

SEMISIMPLE LIE GROUPS SATISFY PROPERTY RD, A SHORT PROOF

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ABSTRACT. We give a short elementary proof of the fact that connected semisimple real Lie groups satisfy property RD. The proof is based on a process of linearization.

1. INTRODUCTION

A length function $L : G \rightarrow \mathbb{R}_+^*$ on a locally compact group G is a measurable function satisfying

- (1) $L(e) = 0$ where e is the neutral element of G
- (2) $L(g^{-1}) = L(g)$
- (3) $L(gh) \leq L(g) + L(h)$.

A unitary representation $\pi : G \rightarrow U(H)$ on a complex Hilbert space has property RD with respect to L if there exists $C > 0$ and $d \geq 1$ such that for each pair of unit vectors ξ and η in H , we have

$$\int_G \frac{|\langle \pi(g)\xi, \eta \rangle|^2}{(1 + L(g))^d} dg \leq C$$

where dg is a (left) Haar measure on G . We say that G has property RD if its regular representation has property RD with respect to L . First established for free groups by U. Haagerup in [6], property RD has been introduced and studied as such by P. Jolissaint in [8], who notably established it for groups of polynomial growth, and for classical hyperbolic groups. See [12][Chap. 8, p.69], and for more details.

If π denotes a unitary representation on a Hilbert space H , then $\bar{\pi}$ denotes its conjugate representation on the conjugate Hilbert space \bar{H} . The process of linearisation consists in working with $\sigma : G \rightarrow U(\bar{H} \otimes H)$ the unitary representation $\sigma = \bar{\pi} \otimes \pi$, see [3][section 2.2].

A connected semisimple real Lie group with finite center can be written $G = KP$ where K is a compact connected subgroup, and P a closed amenable subgroup. We denote by Δ_P the right-modular function of P . Extend to G the map Δ_P of P as $\Delta : G \rightarrow \mathbb{R}_+^*$ with $\Delta(g) = \Delta(kp) := \Delta_P(p)$. It's well defined because $K \cap P$ is compact (observe that $\Delta_P|_{K \cap P} = 1$). The quotient G/P carries a unique quasi-invariant measure μ , such that the Radon-Nikodym derivative at $(g, x) \in G \times G/P$ denoted by

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$c(g, x) = \frac{dg_*\mu}{d\mu}(x)$ with $g_*\mu(A) = \mu(g^{-1}A)$, satisfies $\frac{dg_*\mu}{d\mu}(x) = \frac{\Delta(gx)}{\Delta(x)}$ for all $g \in G$ and $x \in G/P$, (notice that for all $g \in G$, the function $x \in G/P \mapsto \frac{\Delta(gx)}{\Delta(x)} \in \mathbb{R}_+^*$ is well defined). We refer to [2][Appendix B, Lemma B.1.3, p. 344-345] for more details. Consider the quasi-regular representation $\lambda_{G/P} : G \rightarrow U(L^2(G/P))$ associated to P , defined by $(\lambda_{G/P}(g)\xi)(x) = c(g^{-1}, x)^{\frac{1}{2}}\xi(g^{-1}x)$. Denote by dk the Haar measure on K , and under the identification $G/P = K/(K \cap P)$, denote by $d[k]$ the measure μ on G/P .

The well-known Harish-Chandra function is defined by $\Xi(g) := \langle \lambda_{G/P}(g)1_{G/P}, 1_{G/P} \rangle$ where $1_{G/P}$ denotes the characteristic function of the space G/P .

In the rest of the paper we set $\sigma = \overline{\lambda_{G/P}} \otimes \lambda_{G/P}$. Observe that $\overline{L^2(G/P)} \otimes L^2(G/P) \cong L^2(G/P \times G/P)$, via: $\xi \otimes \eta \mapsto ((x, y) \mapsto \overline{\xi(x)}\eta(y))$. Notice that σ preserves the cone of positive functions on $L^2(G/P \times G/P)$.

Let G be a (non compact) connected semisimple real Lie group. Let \mathfrak{g} be its Lie algebra. Let θ be a Cartan involution. Define the bilinear form denoted by (X, Y) such that for all $X, Y \in \mathfrak{g}$, $(X, Y) = -B(X, \theta(Y))$ where B is the Killing form. Set $|X| = \sqrt{(X, X)}$. Write $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{p}$ the eigenvector space decomposition associated to θ (\mathfrak{l} for the eigenvalue 1). Let K be the compact subgroup defined as the connected subgroup whose Lie algebra \mathfrak{l} is the set of fixed points of θ . Fix $\mathfrak{a} \subset \mathfrak{p}$ a maximal abelian subalgebra of \mathfrak{p} . Consider the roots system Σ associated to \mathfrak{a} and let Σ^+ be the set of positive roots, and define the corresponding positive Weyl chamber as

$$\mathfrak{a}^+ := \{H \in \mathfrak{a}, \alpha(H) > 0, \forall \alpha \in \Sigma^+\}.$$

Let $A^+ = Cl(\exp(\mathfrak{a}^+))$, where Cl denotes the closure of $\exp(\mathfrak{a}^+)$. Consider the corresponding polar decomposition KA^+K . Then define the length function

$$L(g) = L(k_1 e^H k_2) := |H|$$

where $g = k_1 e^H k_2$ with $e^H \in A^+$. Notice that L is K bi-invariant. The disintegration of the Haar measure on G according to the polar decomposition is

$$dg = dk J(H) dH dk$$

where dk is the Haar measure on K , dH the Lebesgue measure on \mathfrak{a}^+ , and

$$J(H) = \prod_{\alpha \in \Sigma^+} \left(\frac{e^{\alpha(H)} - e^{-\alpha(H)}}{2} \right)^{n_\alpha}$$

where n_α denotes the dimension of the root space associated to α . See [9][Chap.V, section 5, Proposition 5.28, p.141-142], [5][Chap. 2, §2.2, p.65] and [5][Chap 2, Proposition 2.4.6, p.73] for more details.

The aim of this note is to give a short proof of the following known result ([4],[7]).

Theorem. (*C. Herz.*) *Let G be a connected real semisimple Lie group with finite center. Then G has property RD with respect to L .*

See [4][Proposition 5.5 and Lemma 6.3] for the case G has infinite center.

2. PROOF

Proof. We shall prove that the quasi-regular representation has property RD with respect to L defined above. This implies that the regular representation has property RD with respect to L by Lemma 2.3 in [11]. Write $G = KP$ where K is a compact subgroup and P is a closed amenable subgroup of G . It's sufficient to prove that there exists $d_0 \geq 1$ and $C_0 \geq 0$ such that $\int_G \frac{\langle \lambda_{G/P}(g)\xi, \xi \rangle^2}{(1+L(g))^{d_0}} dg < C_0$, for positive functions ξ , with $\|\xi\| = 1$.

Take $\xi \in L^2(G/P)$ such that $\xi \geq 0$, and $\|\xi\| = 1$. Define the function

$$F : G/P \times G/P \rightarrow \mathbb{R}_+ \\ (x, y) \mapsto \int_K \sigma(k)(\xi \otimes \xi)(x, y) dk.$$

For all $(x, y) \in G/P \times G/P$, we have by the Cauchy-Schwarz inequality:

$$\begin{aligned} \int_K \sigma(k)(\xi \otimes \xi)(x, y) dk &= \int_K \xi(k^{-1}x) \xi(k^{-1}y) dk \\ &\leq \left(\int_K \xi^2(k^{-1}x) dk \right)^{\frac{1}{2}} \left(\int_K \xi^2(k^{-1}y) dk \right)^{\frac{1}{2}}. \end{aligned}$$

Observe that the function $f : x \in G/P \mapsto \int_K \xi^2(k^{-1}x) dk \in \mathbb{R}_+$ is constant. Indeed, fix $x \in G/P$ and let y in G/P . Write $y = hx$ for some $h \in K$ (as K acts transitively on G/P). By invariance of the Haar measure we have $f(y) = \int_K \xi^2(k^{-1}y) dk = \int_K \xi^2(k^{-1}hx) dk = \int_K \xi^2(k^{-1}x) dk = f(x)$. If e is the neutral element in G , we write $[e] \in G/P$. We have for all $x \in G/P$, $f(x) = f([e])$.

Hence, for all $x \in G/P$ we have

$$\begin{aligned} \int_K \xi^2(k^{-1}x) dk &= \int_K \xi^2(k^{-1}[e]) dk \\ &= \int_{K/K \cap P} \xi^2([k^{-1}]) d[k] \\ &= \|\xi\|^2 = 1. \end{aligned}$$

Therefore $\|F\|_\infty := \sup \{F(x, y), (x, y) \in G/P \times G/P\} \leq 1$. Hence $0 \leq F \leq 1_{G/P \times G/P}$, where $1_{G/P \times G/P}$ denotes the characteristic function of $G/P \times G/P$.

Let r be the number of indivisible positive roots in \mathfrak{a} . We know that there exists $C > 0$ such that for all $H \in \mathfrak{a}$ where $e^H \in A^+$ we have

$$\Xi(e^H) \leq C e^{-\rho(H)} (1 + L(e^H))^r$$

with $\rho = \frac{1}{2} \sum_{\alpha \in \Sigma^+} n_\alpha \alpha \in \mathfrak{a}^+$, see [5][Chap 4, Theorem 4.6.4, p.161]. Hence for $d_0 > \dim(\mathfrak{a}) + 2r$, we have

$$\int_{\mathfrak{a}^+} \frac{\Xi^2(e^H)}{(1 + L(e^H))^{d_0}} J(H) dH < \infty.$$

We obtain for all $d \geq 0$ and for all positive functions ξ , with $\|\xi\| = 1$:

$$\begin{aligned}
\int_G \frac{\langle \lambda_{G/P}(g)\xi, \xi \rangle^2}{(1 + L(g))^d} dg &= \int_G \frac{\langle \lambda_{G/P}(g)\xi, \xi \rangle \langle \lambda_{G/P}(g)\xi, \xi \rangle}{(1 + L(g))^d} dg \\
&= \int_G \frac{\langle \sigma(g)\xi \otimes \xi, \xi \otimes \xi \rangle}{(1 + L(g))^d} dg \\
&= \int_K \int_{\mathfrak{a}^+} \int_K \frac{\langle \sigma(k_1 e^H k_2)\xi \otimes \xi, \xi \otimes \xi \rangle}{(1 + L(k_1 e^H k_2))^d} J(H) dk_1 dH dk_2 \\
&= \int_K \int_{\mathfrak{a}^+} \int_K \frac{\langle \sigma(e^H)\sigma(k_2)(\xi \otimes \xi), \sigma(k_1^{-1})(\xi \otimes \xi) \rangle}{(1 + L(e^H))^d} J(H) dk_1 dH dk_2 \\
&= \int_{\mathfrak{a}^+} \frac{\langle \sigma(e^H) \left(\int_K \sigma(k_2)(\xi \otimes \xi) dk_2 \right), \left(\int_K \sigma(k_1^{-1})(\xi \otimes \xi) dk_1 \right) \rangle}{(1 + L(e^H))^d} J(H) dH \\
&= \int_{\mathfrak{a}^+} \frac{\langle \sigma(e^H)F, F \rangle}{(1 + L(e^H))^d} J(H) dH \\
&\leq \int_{\mathfrak{a}^+} \frac{\langle \sigma(e^H)1_{G/P \times G/P}, 1_{G/P \times G/P} \rangle}{(1 + L(e^H))^d} J(H) dH \\
&= \int_{\mathfrak{a}^+} \frac{\langle \lambda_{G/P}(e^H)1_{G/P}, 1_{G/P} \rangle^2}{(1 + L(e^H))^d} J(H) dH \\
&= \int_{\mathfrak{a}^+} \frac{\Xi^2(e^H)}{(1 + L(e^H))^d} J(H) dH
\end{aligned}$$

Take $d_0 > \dim(\mathfrak{a}) + 2r$ and $C_0 = \int_{\mathfrak{a}^+} \frac{\Xi^2(e^H)}{(1 + L(e^H))^{d_0}} J(H) dH$. We have found $d_0 \geq 1$ and $C_0 > 0$ such that for all positive functions ξ in $L^2(G/P)$ with $\|\xi\| = 1$, we have

$$\int_G \frac{\langle \lambda_{G/P}(g)\xi, \xi \rangle^2}{(1 + L(g))^{d_0}} dg \leq C_0$$

as required. \square

Remark 2.1. *The same approach applies to algebraic semisimple Lie groups over local fields. See [1][section 1, (1.3)] and [13][Lemme II.1.5].*

Remark 2.2. *It's not hard to see that this approach shows that the representations of the principal series of G (of class one, see [5] (3.1.12) p.103) satisfy also property RD.*

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REFERENCES

- [1] J. Arthur, *A local trace formula*, Publications mathématiques de l'I.H.É.S., tome 73 (1991), p. 5-96.
- [2] Bachir Bekka, Pierre de la Harpe, and Alain Valette, *Kazhdan's property (T)*, New Mathematical Monographs, vol. 11, Cambridge University Press, Cambridge, 2008.

- [3] A. Boyer, *Quasi-regular representations and property RD*, Preprint, arxiv: 1305.0480, 2013.
- [4] I. Chatterji, C. Pittet, and L. Saloff-Coste, *Connected Lie groups and property RD*, Duke Math. J. 137 (2007), no. 3, 511-536.
- [5] R. Gangolli, V.S. Varadarajan, *Harmonic Analysis of Spherical Functions on Real Reductive Groups*, Springer-Verlag, New-York, 1988.
- [6] U. Haagerup, *An example of a nonnuclear C^* -algebra which has the metric approximation property*, Invent. Math. 50 (1978/79), no. 3, 279-293.
- [7] C. Herz, *Sur le phénomène de Kunze-Stein*, C. R. Acad. Sci. Paris Sr. A-B 271 (1970), A491-A493.
- [8] P. Jolissaint, *Rapidly decreasing functions in reduced C^* -algebras of groups*, Trans. Amer. Math. Soc. 317 (1990), no. 1, 167-196.
- [9] A-W. Knap, *Representation theory of semisimple groups*, Princeton landmarks in mathematics, 2001.
- [10] M. Perrone, *Rapid decay and weak containment of unitary representations*, unpublished notes, 2009.
- [11] Y. Shalom, *Rigidity, unitary representations of semisimple groups, and fundamental groups of manifolds with rank one transformation group*, Ann. of Math. (2) 152 (2000), no. 1, 113-182.
- [12] A. Valette, *Introduction to the Baum-Connes conjecture*, Lectures in Mathematics ETH Zurich. Birkhauser Verlag, Basel, (2002).
- [13] J-L. Walspurger, *La formule de Plancherel pour les groupes p -adiques. D'après Harish-Chandra*, Journal of the Institute of Mathematics of Jussieu, Volume 2, Issue 02. April 2003, pp 235-333.

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